Van der Pol Equation: Overview, Derivation, and Examination of Solutions

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What is the Van der Pol equation?

The **Van der Pol equaion** is a second order nonlinear differential equation and an ordinary differential equation with nonlinear damping. It is defined as:

$$y'' - \mu(1 - y^2)y' + y = 0$$

Where μ is a constant that determines the strength of the damping on the oscillations.

Some cool facts:

- The Van der Pol equation can be thought of as a simple linear harmonic oscillator with nonlinear damping. How do we know this? Let's set $\mu = 0$ and see:

  $$y'' - 0(1 - y^2)y' + y = 0$$
  $$y'' + y = 0$$

  This DE is the same simple linear harmonic oscillator whose solutions are $c(x)$ and $s(x)$. We proved in class that these functions share the same properties of sine and cosine and in fact are sine and cosine. Thus, the Van der Pol equation is simply a generalization of sine and cosine. Cool!

- The value y is equal to its derivative for this oscillator, $y' = y$. We will see why in the derivation and come back to it in the examination of solutions.

- This equation describes self-sustaining oscillations for which energy is fed into smaller oscillations and removed from larger oscillations in a damping process. This description will become more clear in the solutions demonstrated later.

Derivation and Theory

The basic assumption by looking at the definition of the Van der Pol equation is that it is a nonlinear second order differential equation. In its most general form:

$$y'' + F(y') + y = 0$$

A more specific form of this generalization is the definition of the Van der Pol equation:

$$y'' - \mu(1 - y^2)y' + y = 0$$

This form of a nonlinear second order DE can be derived from another equation called the **Rayleigh differential equation**.

The Rayleigh differential equation is defined as:

$$y'' - \mu(1 + \frac{1}{3}y'^2)y' + y = 0$$

The equation was discovered by a man named Lord Rayleigh in the late-1800s and is used to describe certain dynamic, unstable physical processes. Some notable applications include:

- The analysis of simple distillation in chemistry. Since the process of distillaion is naturally unstable, the equation was derived on a differential approach to changes in concentration of solution with time.
- In fluid mechanics, a special case of the equation can describe the dynamics of a spherical bubble in an infinite body of water.
- Auto-oscillations of sound vibrations in acoustics.

Van der Pol discovered that if he took the derivative of the Rayleigh differential equation and set $y' = y$, a new equation would arise that describes an entirely new world of ocsillations. See the operations below:

$$y'' - \mu(1 + \frac{1}{3}y'^2)y' + y = 0$$

Expand the parentheses:

$$y'' - \mu y' - \mu\frac{1}{3}y'^3 + y = 0$$

Take the derivative:

$$\frac{d}{dy}(y'' - \mu y' - \mu\frac{1}{3}y'^3 + y) = 0$$

We get:

$$y''' - \mu y'' - \mu y'^2 y'' + y' = 0$$

Now, we make it look like the Van der Pol equation we know:

$$y''' - \mu(y'' - y'^2 y'') + y' = 0$$

$$y''' - \mu(1 - y'^2)y'' + y' = 0$$

This looks awfully close to our original definition of the Van der Pol equation. Now just use one of our "cool facts" from above, $y' = y$ and we get:

$$y'' - \mu(1 - y^2)y' + y = 0$$

Another cool fact arising from this derivation: If we define the function y to be equal to its derivative, what function satisfies this property? There is only one function: $y = Ae^x$, where A is any constant. Thus, exponential and log properties are also used in the definition of these Van der Pol oscillations.

The Van der Pol oscillator can also be evaluated as a system of differential equations. By setting the equation up in the following manner, we can get a general idea of how the solutions of the equation will look. Let

$$y = x' + \frac{1}{3}x^3 - x$$

Take the derivative:

$$y' = x'' + x^2x' - x' = -x$$

Now, we have enough information to create a system of differential equations:

$$x' = y - \frac{1}{3}x^3 + x$$

$$y' = -x$$

This system is nonlinear since it contains a cubic term, but we can accurately approximate this system on a smaller scale by looking at the linear parts near the origin. The solutions near the origin of this system can be expressed in terms of a matrix M:

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

Calculating the eigenvalues of M will reveal some interesting characteristics about the oscillations of these solutions. The characteristic equation for the 2x2 matrix M is:

$$\lambda^2 - Trace(A)\lambda + Det(A)$$

$$\lambda^2 - \lambda + 1$$

So λ is equivalent to

$$\lambda = \frac{-(-1)^2 \pm \sqrt{(-1)^2 - 4(1)}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

These two values of λ are eigenvalues for the matrix M which reveal some interesting facts about the solutions for this system.

- The eigenvalues are complex, so the imaginary portion reveals that solutions to the system will oscillate.
- The real part of the eigenvalues is positive, so the solutions will spiral outward.

Exploring Solutions

The derivation above clears much up about what makes up the core of the Van der Pol equation mathematically, but several questions still remain:

- Can we calculate numerically the solutions of the Van der Pol equation?

No. These solutions cannot be calculated numerically. Visualizations are the best way to understand and describe the solutions to this equation.

- How can we be sure these solution curves oscillate and never fall off or explode up to $-\infty$ or ∞?

Let's calculate the nullcline of the system. Our equation is

$$x' = y - \frac{1}{3}x^3 + x$$

Which is equivalent to

$$y = \frac{1}{3}x^3 - x + x'$$

Setting $x' = 0$ (which is the definition of a nullcline) we get

$$y = \frac{1}{3}x^3 - x$$

Setting $y' = 0$ for our other equation $y' = -x$, we get

$$x = 0$$

Now, let's plot these two nullclines. Figure 1 comes from Dan Flath's notes on the Van der Pol equation.

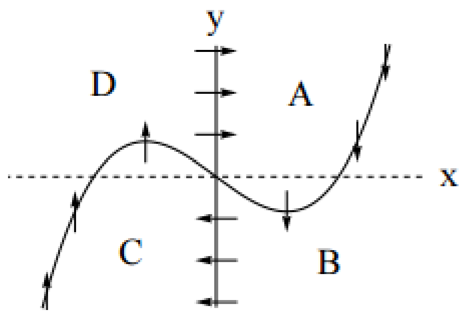

Figure 1: Nullclines for VDP Equation (Flath)

So how does this help us prove that the solutions oscillate? Let's examine some facts about these solutions that will help prove their oscillatory nature:

1. Any solution touching the cubic nullcline $y = \frac{1}{3}x^3 - x$ has no x change ($x' = 0$). This means wherever a solution curve is touching this nullcline, the solution curve must be vertical at that point. Above this nullcline, the change in x is positive. Below this nullcline, the change in x is negative.
2. Any solution touching the vertical line $x = 0$ has no y change ($y' = 0$). This means that wherever a solution curve is touching this nullcline, the solution curve must be horizontal at that point. To the left of this nullcline, the change in y is positive. To the right of this nullcline, the change in y is negative.

These facts are accurately represented in Figure 1. Let's track the trajectory of a solution curve using these facts. We will start in region A from Figure 1.

Region A

- x' is positive
- y' is negative

From region A, we see that the solution curves move down and to the right given the positive change in x and negative change in y.

Region B

- x' is negative
- y' is negative

In region B, we see that the solution curves move down and to the left given the two negative derivatives. How do we know that these solution curves don't fall down to $-\infty$? We can briefly prove this using our system of equations defined previously.

$$y' = -x$$

$$x' = y - \frac{1}{3}x^3 - x$$

By flipping the signs of both derivatives, we can get a friendly slope to help with the proof. The definition of slope is $\frac{rise}{run}$, thus we can define the slope of the system as:

$$\frac{y'}{x'} = \frac{x}{\frac{1}{3}x^3 - x - y}$$

The solution curves will never fall to negative infinity ($y = -\infty$) unless its slope approaches negative infinity. Since it is impossible for the slope to reach $-\infty$, or ∞ for that matter, because we have a larger exponent on x in the denominator and no y in the numerator, the solution curves in region B will always intersect the y axis, a nullcline, at some point. Since they intersect the nullcline which has no change in y, the curves will be horizontal at that point and move into region C. The situation for regions A and B is illustrated well in Figure 2, also taken from Dan Flath's notes on the Van der Pol equation.

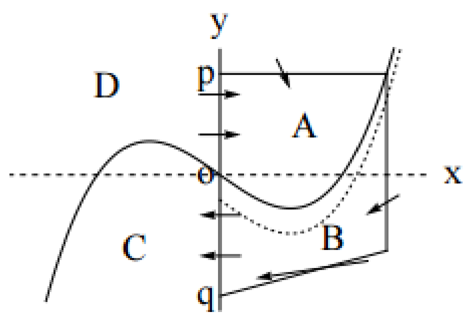

Figure 2: Directions for solution curves in regions A and B (Flath)

Region C

- x' is negative
- y' is positive

Once in region C, we see that the solution curves have a negative change in x and a positive change in y, resulting in movement up and to the left. This upward and left movement will result in an intersection with the cubic nullcline, where all solution curves are vertical due to there being no change in x along that nullcline. This intersection will result in movement into region D.

Region D

- x' is positive
- y' is positive

In region D, the change in x and y are both positive, resulting in movement up and to the right. This region is the other area where concerns might exist about the solution curves exploding to $y = \infty$. It was proven in the section about region B that the change in the y direction cannot outpace the change in the x direction, thus no solution curve will explode to ∞ in region D. Thus, every solution curve must at some point cross the positive y-axis, where each curve will be horizontal since there is no change in y at that nullcline. This results in the solution curves returning to region A.

The tracing of the solution curves using nullclines and slope in each region divided by the nullclines has proven that the solutions to the Van der Pol equation are always oscillations. We can investigate whether these oscillations spiral inward or outward using a **Poincare map**.

In general, a Poincare map illustrates the transformation of a point to another point. In this specific context, it demonstrates the location of a solution curve when it first intersects the y-axis to the next time it intersects the y-axis. The driving factor behind the transformation is determined by the solution curve itself. See the illustration in Figure 3 below of a Poincare map from Dan Flath's notes on the Van der Pol oscillator.

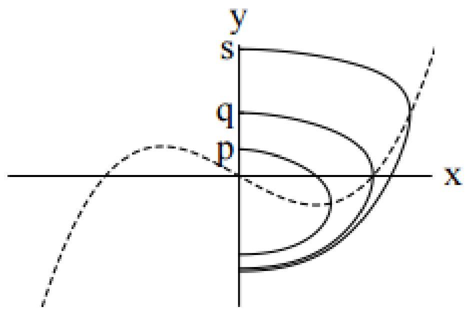

Figure 3: Poincare Map illustration for VDP equation (Flath)

We can see from the illustration that the order of magnitude of the intersection locations for points p, q and s are reversed for their Poincare transformation. That is, $p < q < s$ and $P(s) < P(q) < P(p)$. More importantly, a few rules can be defined by the Poincare location of our points on the solution curves:

1. Solution curve spirals out if $|p| < |P(p)|$ for any point p.
2. Solution curve spirals in if $|p| > |P(p)|$ for any point p.
3. Solution curve is closed if $|p| = |P(p)|$ for any point p.

Every solution curve oscillates until it eventually becomes closed. This will be illustrated in the following final section.

- What do the solutions look like and how does altering the parameter μ affect these solutions?

As mentioned previously, the solutions to the Van der Pol equation are all oscillations. This means that their solutions in the phase diagrams will oscillate based on the parameter μ and eventually stabilize in a closed, periodic manner. This makes sense, as we can recall from earlier in this document that the Van der Pol equation is a generalization of the differential equation whose solutions describe two classically stable periodic functions, sine and cosine.

The following solution curves are all plotted using initial conditions $x(0) = 0$ and $x'(0) = 1$ unless otherwise stated. Let's investigate a solution on the phase plane for $\mu = 0$.

See Figure 4 on the following page.

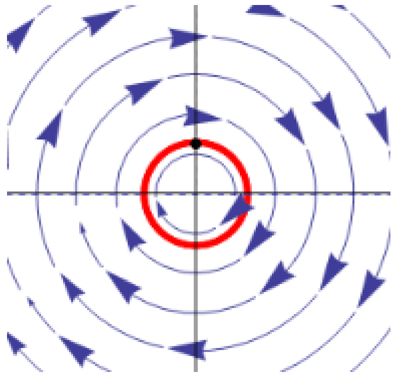

Figure 4: Solution to VDP equation with $\mu = 0$

The solution curve is completely circular and following the circular slope field lines. The oscillation is closed, periodic at 2π, and has magnitide 1. This perfectly describes the sine and cosine curves, and in fact sine and cosine are the two solutions to $y'' + y = 0$ as we proved in class.

What if μ gets bigger?

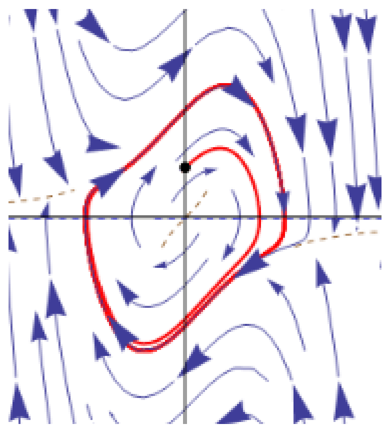

Figure 5: Solution to VDP equation with $\mu = 1$

We see the oscillation is no longer closed and spirals outward until it becomes closed. The dotted line running through the origin represents the cubic nullcline that we used to evaluate the oscillations earlier.

What if we keep $\mu = 1$ and change the initial condition to outside the stable point of oscillation where solution curves become closed?

See Figure 6 on following page.

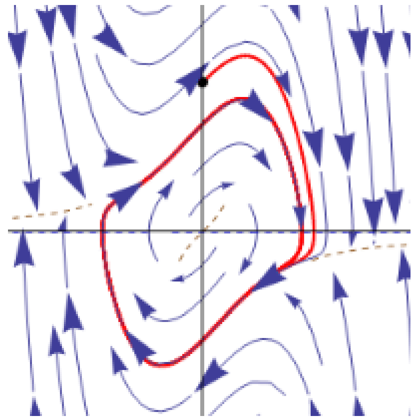

Figure 6: Solution to VDP equation with $\mu = 1$ with initial condition $x(0) = 0$ and $x'(0) = 3$

Just as expected and as illustrated in the Poincare map, the solution curve now spirals inward instead of outward. Amazing how a simple shift in initial conditions can reverse the direction of spiraling for a solution curve!

Let's make μ bigger by a few multiples and see what happens.

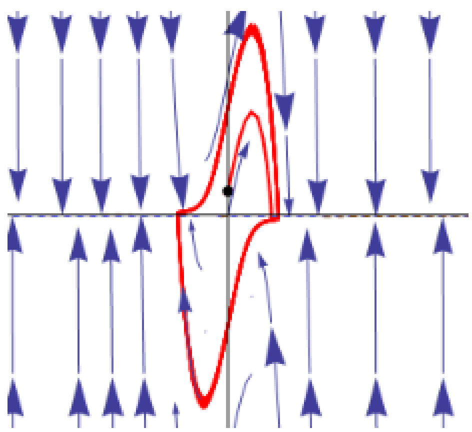

Figure 7: Solution to VDP equation with $\mu = 5$

It becomes easy to see the effect of increasing μ even after only a few additional plots. It affects the nonlinearity of the solution curve and determines the strength of the damping (higher μ, longer oscillation), which in turn lengthens the limit cycle of the solution curve. This concept is clearly illustrated in a figure posted on Wikimedia about the Van der Pol oscillator. See the figure on the following page.

It can be determined from merely looking at the solution to the Van der Pol equation that increasing μ lengthens the limit cycle of a solution curve, thus making its oscillations more stiff and the period of the solution larger.

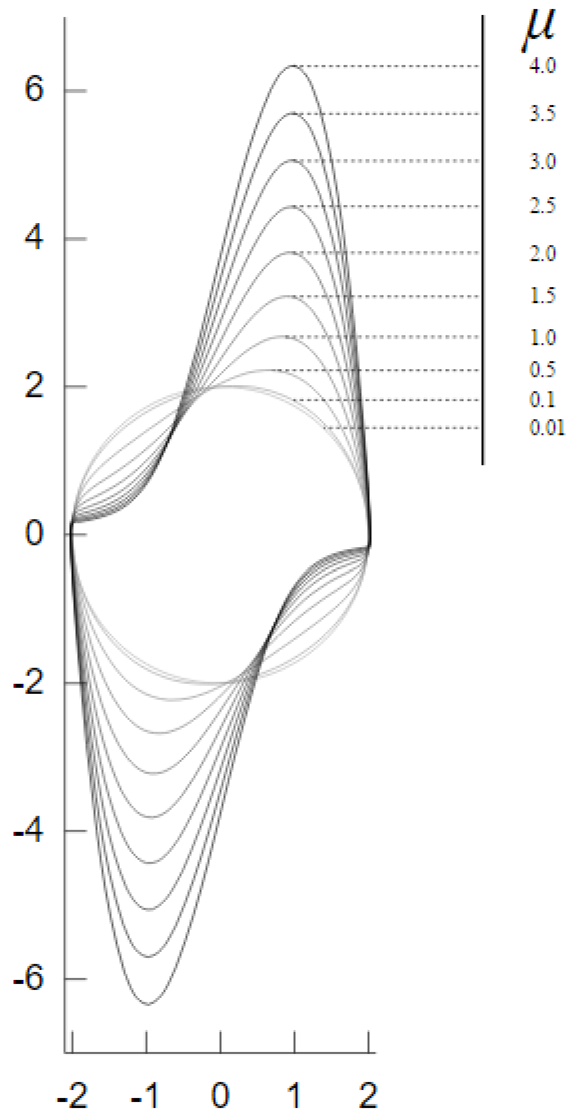

Figure 8: Solution curves to VDP equation with varying μ (Image: Wikimedia)

In summary, this project has taught me an unbelievable amount about the power of the Van der Pol equation and, more broadly, differential equations. There are so many ways to describe and model elements of the world around us if one can abstractly think up a system of equations that could describe a situation. Van der Pol's generalization of sine and cosine serve as a bridge to using familiar oscillations to describe complex, abstract, and otherwise unmodelable situations.

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